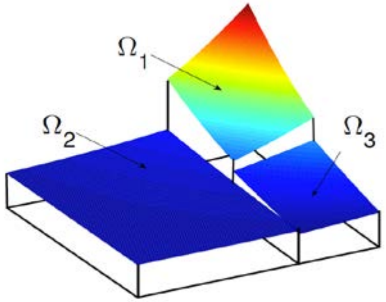
Introduction to high-order DG method

Governing equations: $$\frac{\partial \mathsf{U}}{\partial t} + \nabla \cdot \mathsf{F} = \nabla \cdot \mathsf{Q}$$



FD

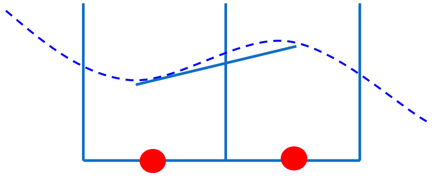
$$\frac{(F_{i+1} - F_i)}{\Delta x}$$

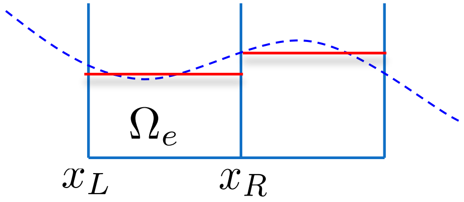
FV

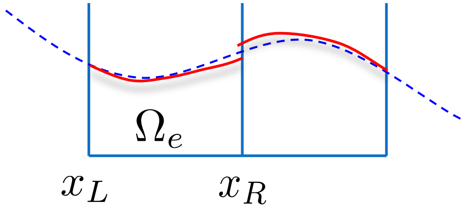
$$\int_{\Omega_e} \partial_x \mathsf{F} dx$$
$$= \mathsf{F}(x_R) - \mathsf{F}(x_L)$$
$$\approx \widehat{F}(x_R) - \widehat{F}(x_L)$$

DG

$$\int_{\Omega_e} \phi \partial_x \mathsf{F} dx$$
$$= (\phi \mathsf{F})|_{x_L}^{x_R} - \int_{\Omega_e} \mathsf{F} \partial_x \phi dx$$
$$\approx (\phi \widehat{F})|_{x_L}^{x_R} - \int_{\Omega_e} F \partial_x \phi dx$$



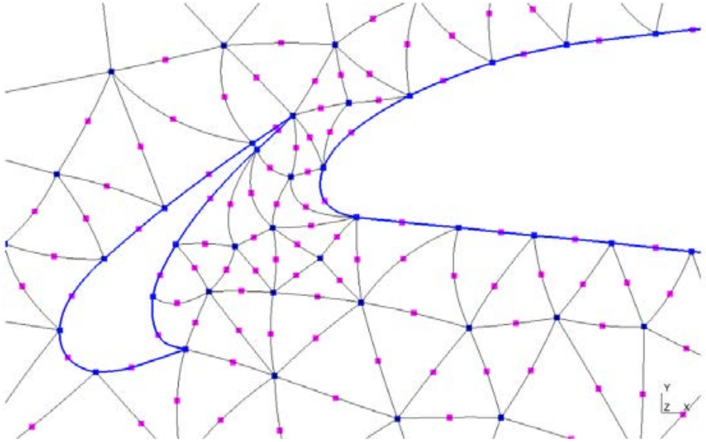


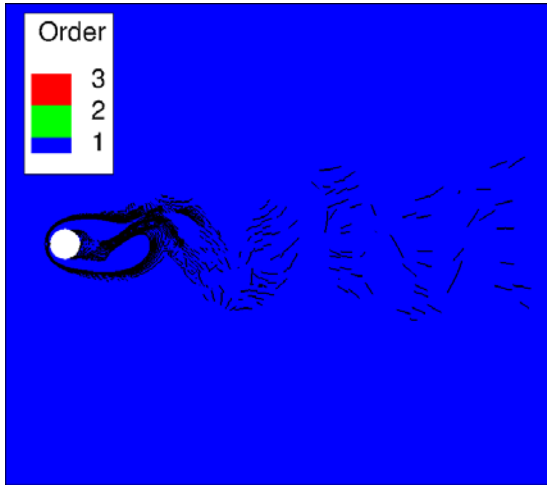


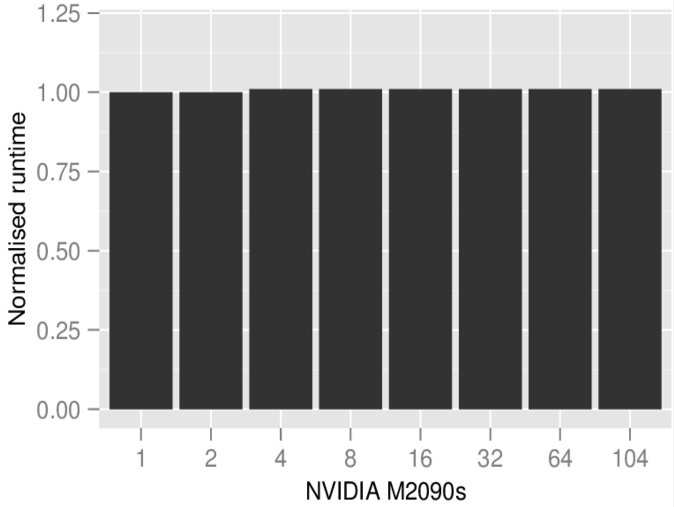
Introduction to high-order DG method

Advantages:

- Compactness
- Superconvergence
- Energy stability
- Adaptivity
- Real-world geometry
- Data locality for HPC







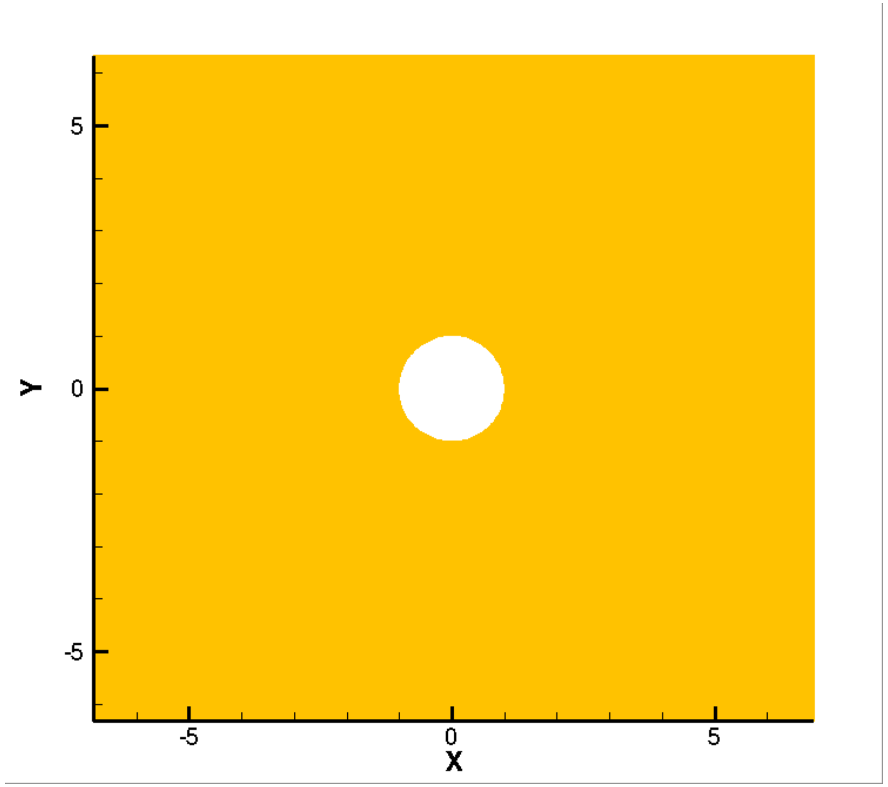
Discontinuous schemes:

- discontinuous Galerkin (Reed & Hill, 1973)
- flux reconstruction (Huynh, 2007)
- discontinuous spectral element (Kopriva, 2002)
- spectral difference (ZJ Wang, 2006)

...

Robustness issue of high-order DG scheme

Illustration of the robustness issue in DG:



DGP4 solution of Mach 0.38 inviscid flow over cylinder

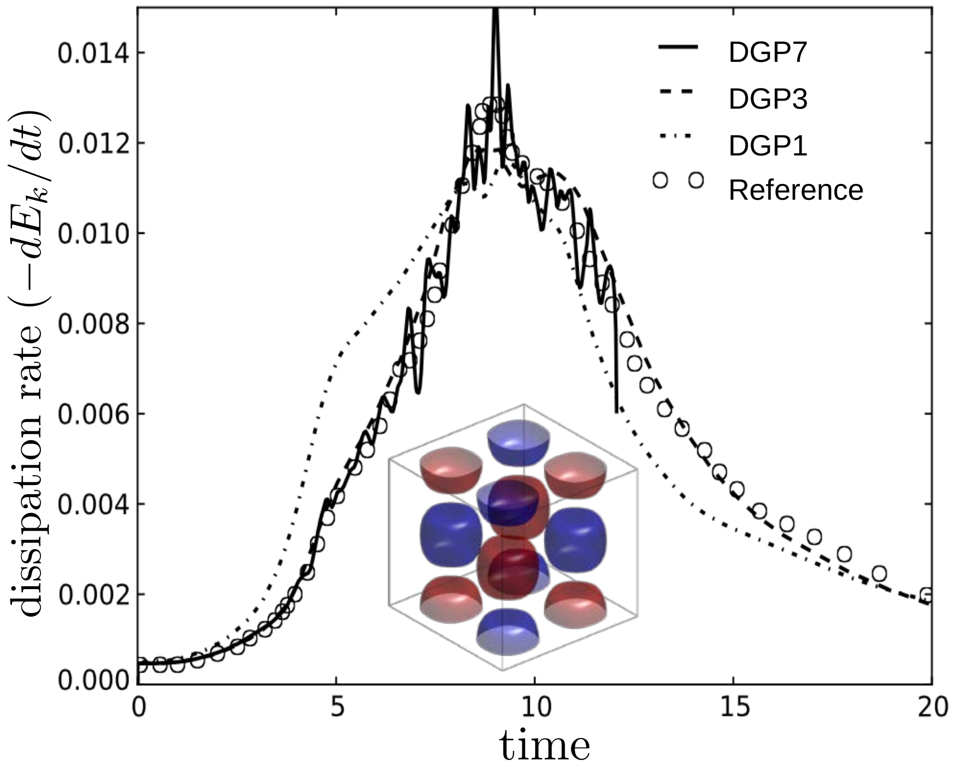
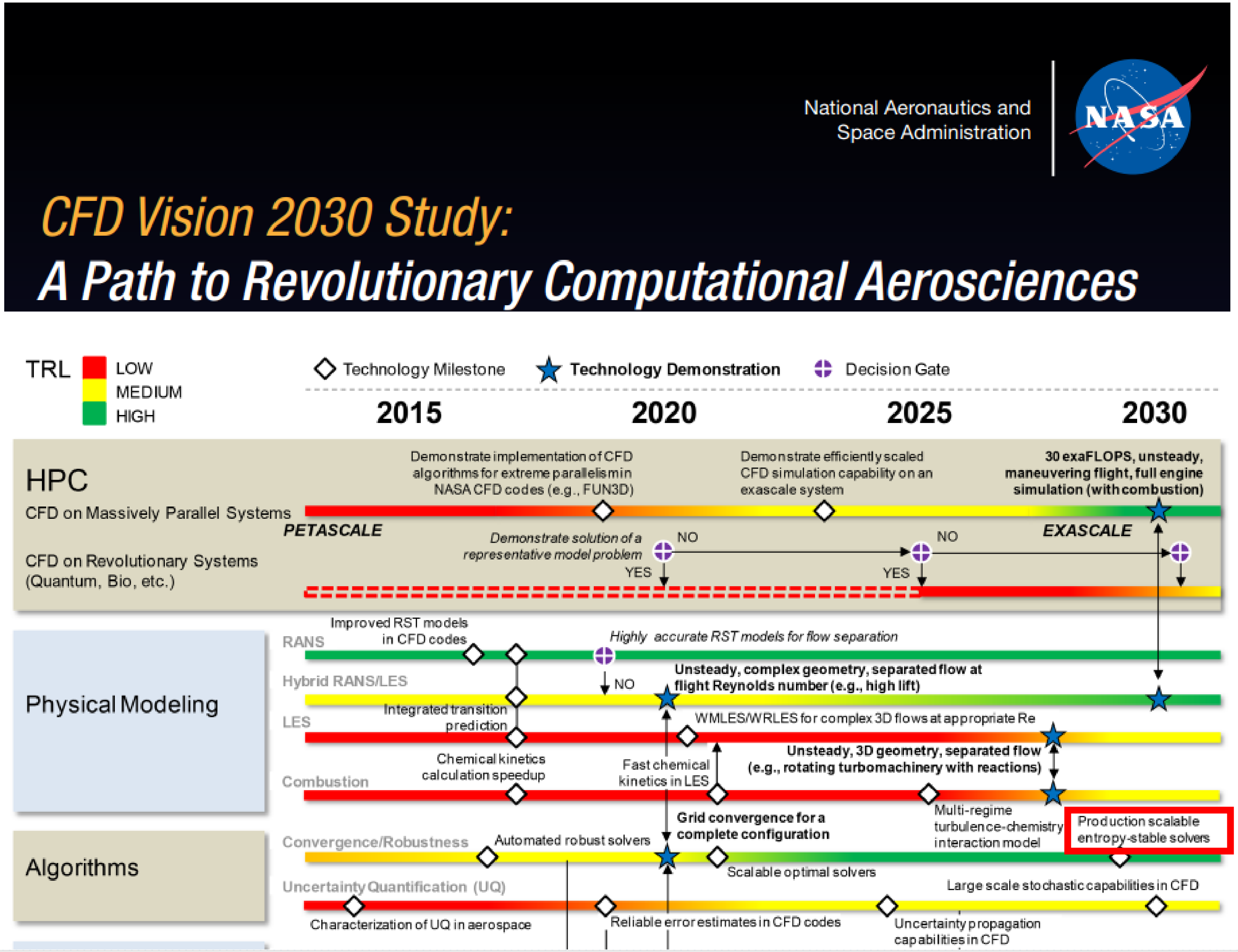


Figure credits: first high-order workshop, 2012.
Bull & Jameson, *AIAA J.*, 2015.

Need of a robust and high-order scheme



"Longer term, high-risk research should focus on the development of truly enabling technologies such as monotone or entropy stable schemes in combination with innovative solvers…"

"Toward the 2030 time frame, it is anticipated that novel entropy stable formulations will begin to bear fruit for industrial simulations"

Entropy-bounded DG scheme

- Governing equations:

$$\partial_t \mathsf{U} + \nabla \cdot \mathsf{F} = 0$$
$$\mathsf{U} = [\rho,\ \rho u,\ \rho E,\ \rho Y]^T$$
$$\mathsf{F} = \left[\rho u,\ \rho u^T u + p\mathbf{I},\ u(\rho E + p),\ \rho u Y\right]^T$$

- Entropy function: $s = \log(p/\rho^\gamma)$, quasi-concave function of U

- Physical principles:
  1. $p > 0$, concave function of U
  2. $\rho > 0$, convex/concave function of U
  3. $s \geq s_0$, s_0 is the minimum entropy on a domain Ω over a finite time internal $[0, \Delta t]$ (the 2nd law of thermodynamics)
  4. $Y \in [0,\ 1]$

Entropy-bounded DG scheme

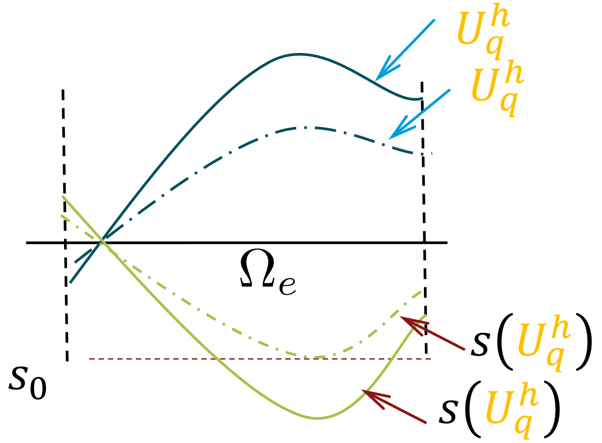
Key idea:

- Impose the inequality physical principles on discrete solutions to ensure "physical realizability"

$$\begin{cases} \rho(U_q^h) \geq 0 \\ p(U_q^h) \geq 0 \\ s(U_q^h) \geq s_0 \end{cases}$$

Key algorithmic ingredients:

1. Limiter to enforce $s(U_q^h) \geq s_0$
2. Sufficient condition to guarantee $s(\overline{U}_e^h) \geq s_0$



Zhang & Shu, *J. Comput. Phys.*, 2009, 2010
Lv & Ihme, *J. Comput. Phys., 2015*

Entropy-bounded DG scheme

Limiter:

- Linear-scaling of elementwise local solutions
- Exploit Jensen's inequality based on convexity/concavity

Define ${}^{\text{L}}U_e = U_e + \epsilon(\overline{U}_e - U_e)$ and given $s(\overline{U}_e) \geq s_0$,
find the smallest ϵ, such that $\forall x_q$,

1. $\rho\left({}^{\text{L}}U_e(x_q)\right) > 0$
2. $s\left({}^{\text{L}}U_e(x_q)\right) \geq s_0$

Apply Jensen's inequalities to expand (*1*) and (*2*):

1. $(1-\epsilon)\rho\left(U_e(x_q)\right) + \epsilon\rho\left(\overline{U}_e(x_q)\right) > 0$
2. $p\left({}^{\text{L}}U_e(x_q)\right) \geq \exp(s_0)\,\rho^{\gamma}\left({}^{\text{L}}U_e(x_q)\right)$

$$(1-\epsilon)p\left(U_e(x_q)\right) + \epsilon p\left(\overline{U}_e(x_q)\right) \geq \exp(s_0)\left[(1-\epsilon)\rho^{\gamma}\left(U_e(x_q)\right) + \epsilon\rho^{\gamma}\left(\overline{U}_e(x_q)\right)\right]$$

Note: $s_0 \to -\infty$, this becomes PP limiter

Entropy-bounded DG scheme

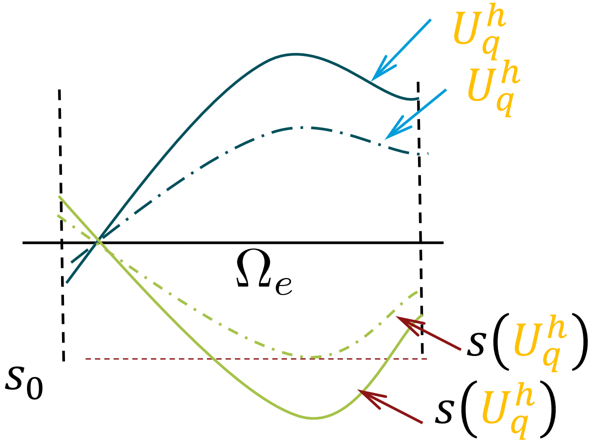
Key idea:

- Impose the inequality physical principles on <u>discrete</u> solutions to ensure "physical realizability"

$$\begin{cases} \rho\left(U_q^h\right) \geq 0 \\ p\left(U_q^h\right) \geq 0 \\ s\left(U_q^h\right) \geq s_0 \end{cases}$$

Key algorithmic ingredients:

1. Limiter to enforce $s\left(U_q^h\right) \geq s_0$
2. Sufficient condition to guarantee $s\left(\overline{U}_e^h\right) \geq s_0$



Zhang & Shu, *J. Comput. Phys.*, 2009, 2010
Lv & Ihme, *J. Comput. Phys., 2015*

Entropy-bounded DG scheme

The cell-average of DG solution after one time-step

$$\overline{U}_e^{t+\Delta t} = \sum_q w_q U_e(x_q)$$

$$\overline{U}_e^{t+\Delta t} = \overline{U}_e - \frac{\Delta t}{h}\left(\hat{F}(U_e(x_l),\ U_{e-1}(x_l),\ -\vec{n}) + \hat{F}(U_e(x_r),\ U_{e+1}(x_r),\ \vec{n})\right)$$

$$= \sum_q \left(w_q - \beta_l \phi_q(x_l) - \beta_r \phi_q(x_r)\right) U_e(x_q)$$

$$+\beta_l U_e(x_l) - \frac{\Delta t}{h}\left(\hat{F}(U_e(x_l),\ U_{e-1}(x_l),\ -\vec{n}) + \hat{F}(U_e(x_l),\ U_e^\star,\ \vec{n})\right)$$

$$+\beta_r U_e(x_r) - \frac{\Delta t}{h}\left(\hat{F}(U_e(x_r),\ U_e^\star,\ -\vec{n}) + \hat{F}(U_e(x_r),\ U_{e+1}(x_r),\ \vec{n})\right)$$

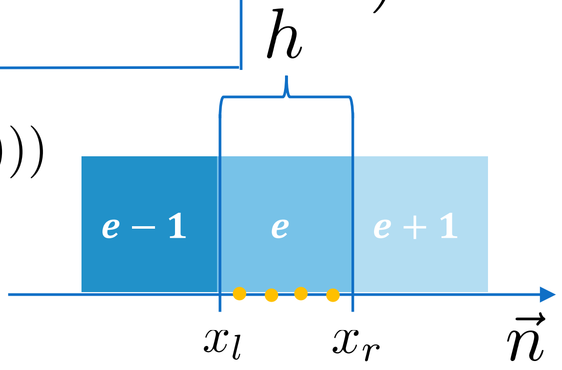
$$U_e(x_{l,r}) = \sum_q \phi_q(x_{l,r}) U_e(x_q)$$

new CFL numbers

$$\frac{\beta_l \Delta t}{h} \quad \frac{\beta_r \Delta t}{h}$$

- $$U_e^\star = \frac{1}{2}\left(U_e(x_l) + U_e(x_r)\right) - \frac{1}{2\lambda}\left(F(U_e(x_r)) - F(U_e(x_l))\right)$$

This is a solution of *Lax* Scheme
(P. D. Lax. Contributions to Nonlinear Functional Analysis, pages 579 603–634. Academic Press, New York, London, 1971.)



Entropy-bounded DG scheme

Three-point FVM system:

- The temporally-updated first-order solution written as

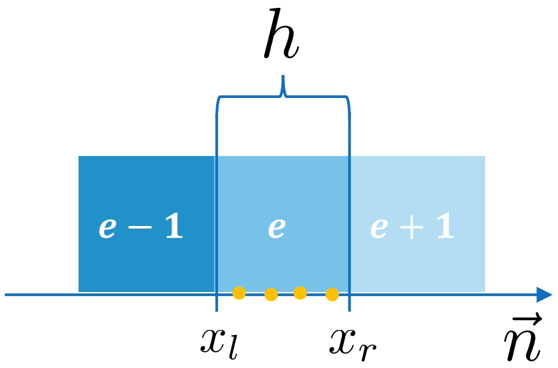
$$\tilde{U}_e^{t+\Delta t} = \tilde{U}_e - \frac{\Delta t}{h}\left(\hat{F}(\tilde{U}_e,\ \tilde{U}_{e-1},\ -\vec{n}) + \hat{F}(\tilde{U}_e,\ \tilde{U}_{e+1},\ \vec{n})\right)$$

Lemma: $s(\tilde{U}_e^{t+\Delta t}) \geq \min_{k\in\{e,\ e\pm1\}} (s(\tilde{U}_k)) = s_0$

1. if $\hat{F}(U_+,\ U_-,\ \vec{n})$ is an entropy-stable flux (e.g., LF flux)
2. CFL condition $\Delta t\lambda/h \leq \alpha$

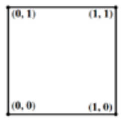
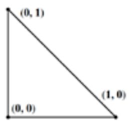
Proof:

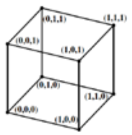
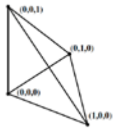
- 1D version: E. Tadmor, *Appl. Numer. Math.*, 1986
- 3D version: Y. Lv & M. Ihme, *J. Comput. Phys.*, 2015



Entropy-bounded DG scheme

- CFL obtained for different element shapes

Element	Order	QR on $\partial\Omega_e$	QR on Ω_e	CFL^{EB}
	$p=1$	/	3	0.5
	$p=2$	/	5	0.167
	$p=3$	/	7	0.123
	$p=4$	/	9	0.073
	$p=1$	3	3	0.25
	$p=2$	5	5	0.083
	$p=3$	7	7	0.062
	$p=4$	9	9	0.036
	$p=1$	3	4	0.135
	$p=2$	5	5	0.067
	$p=3$	7	8	0.058
	$p=4$	9	9	0.033

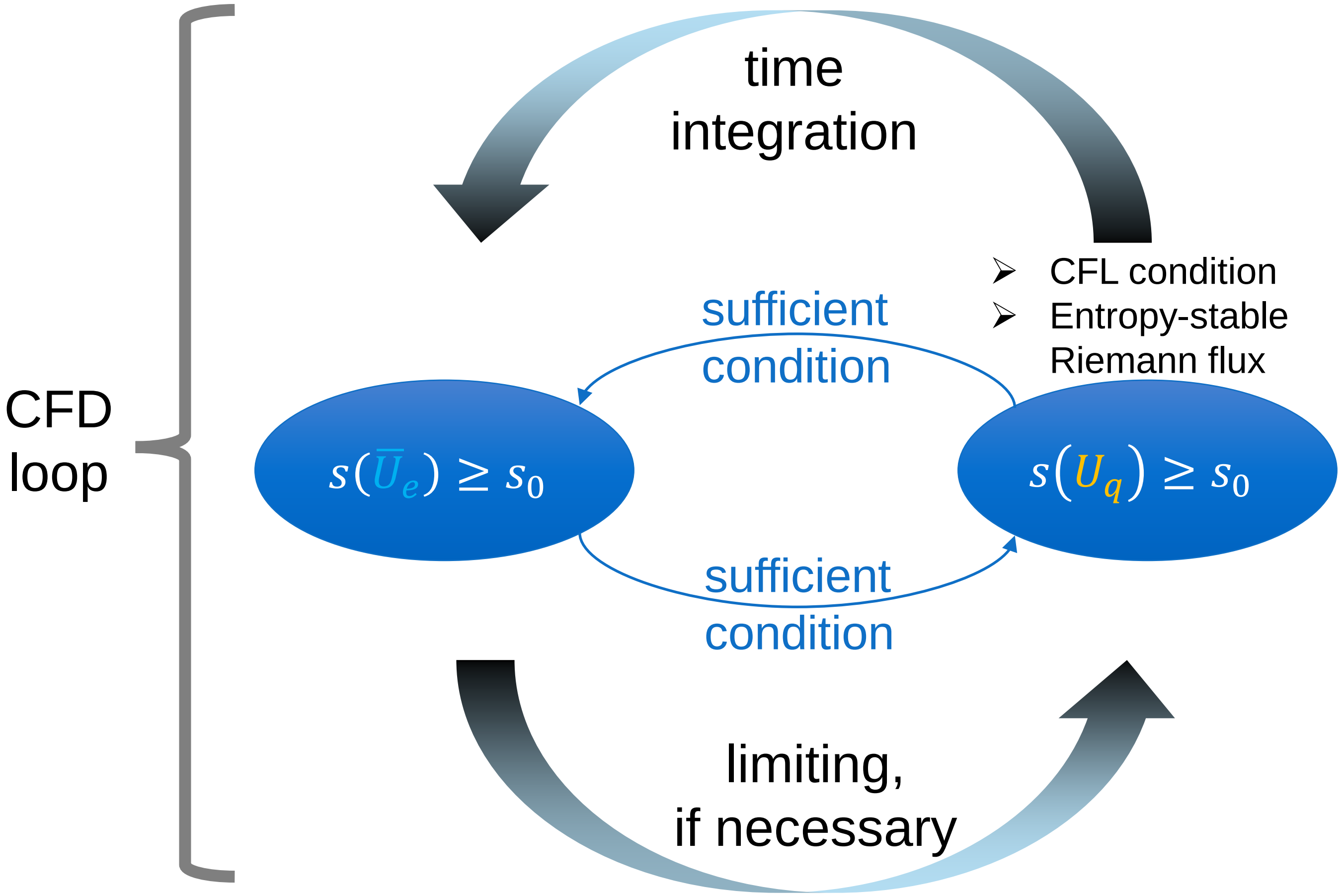
Element	Order	QR on $\partial\Omega_e$	QR on Ω_e	CFL^{EB}
	$p=1$	3	3	0.167
	$p=2$	5	5	0.056
	$p=3$	7	7	0.041
	$p=4$	9	9	0.024
	$p=1$	4	3	0.066
	$p=2$	5	5	0.035
	$p=3$	8	7	0.015
	$p=4$	9	9	0.013

Note: Quadrature rule (QR) applied: Line, Quadrilateral and Brick: tensor-product Gauss-Legendre; Triangle: Dunavant; Tetrahedron: Zhang, et al.

- CFL number can be found for high-order curved elements, which will be element-specific.

Lv & Ihme, *J. Comput. Phys., 2015*

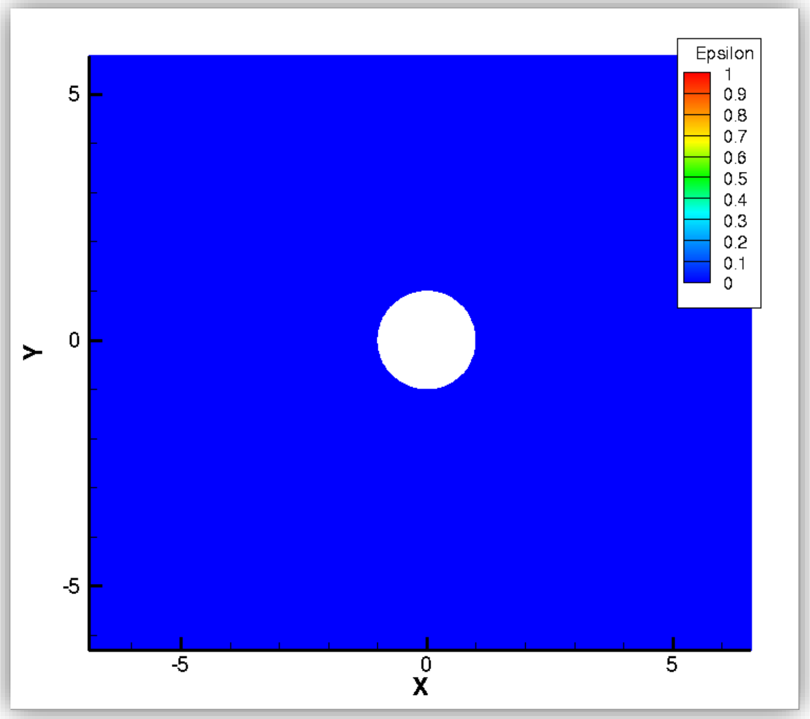
Entropy-bounded DG scheme

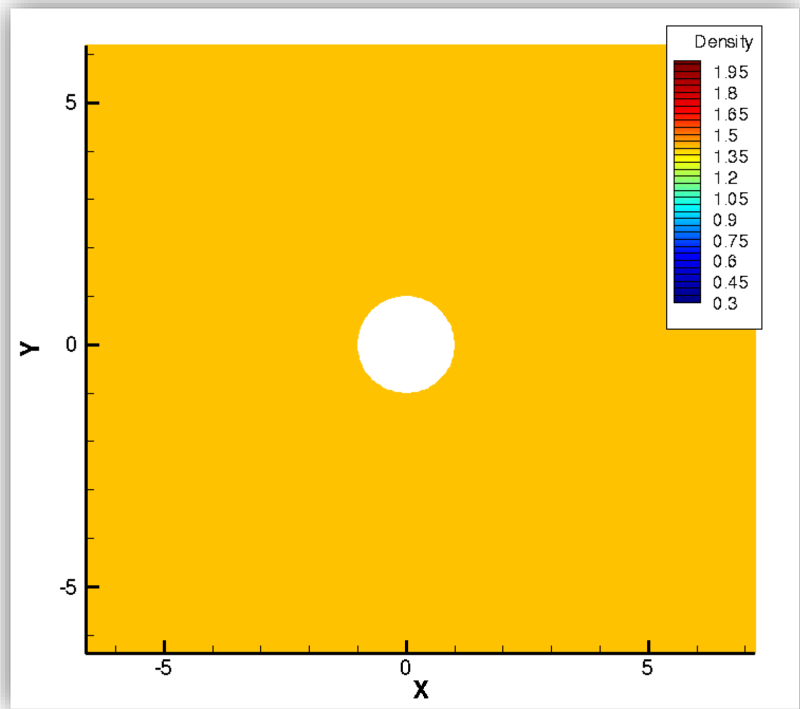


Entropy-bounded DG scheme

Finding 1:

EBDG prevents non-physical solutions during transient stages





Entropy-bounded DG scheme

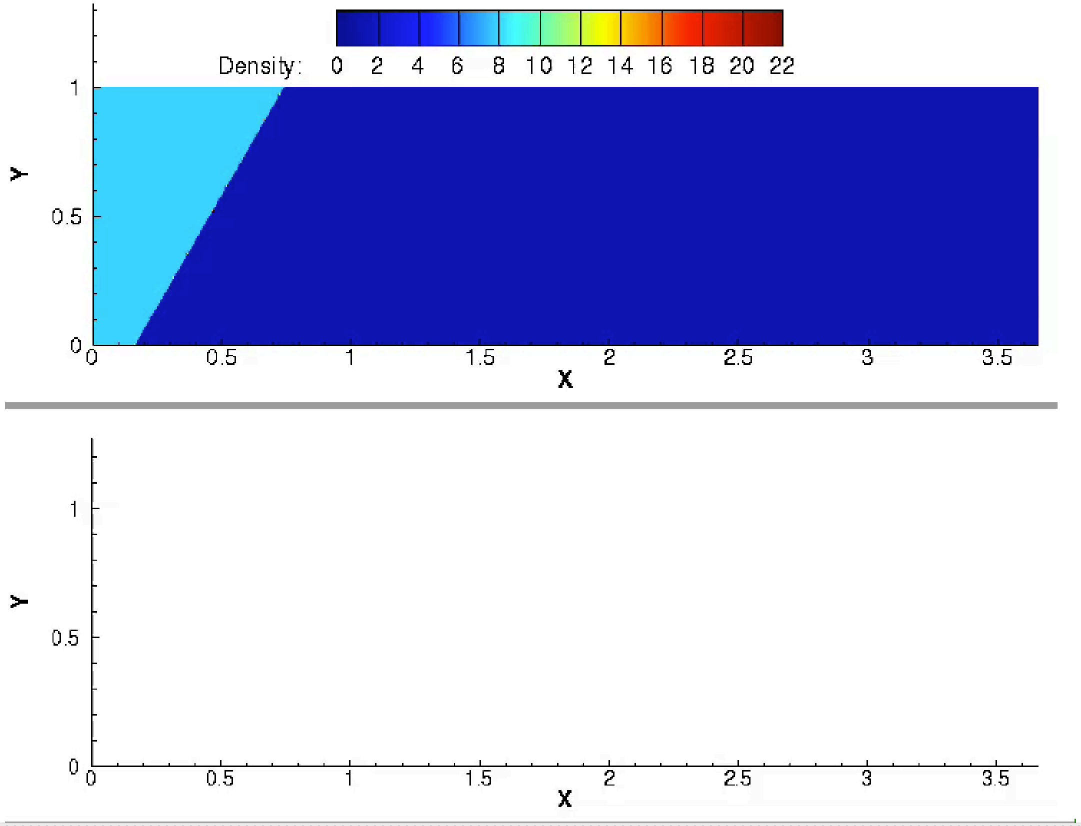
Finding 2:

EBDG preserves the optimal convergence for smooth solutions

Mesh	DGP1		DGP2		DGP3	
	L_2-error	rate	L_2-error	rate	L_2-error	rate
Quadrilateral Elements						
Level 1	7.272e-2	-	1.694e-2	-	3.816e-3	-
Level 2	1.318e-2	2.464	7.219e-4	4.552	1.827e-4	4.384
Level 3	2.441e-3	2.433	6.029e-5	3.582	1.036e-5	4.141
Triangular Elements						
Level 1	1.137e-1	-	2.590e-2	-	4.086e-3	-
Level 2	1.865e-2	2.608	8.899e-4	4.863	1.291e-4	4.984
Level 3	3.391e-3	2.459	7.222e-5	3.623	6.939e-6	4.217

Finding 3:

EBDG ensures nonlinear stability for arbitrary flow conditions and/or mesh configurations



Solution

Trouble-cell marker